Polynomials with exponents in compact convex sets and associated weighted extremal functions

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Seminar on Methods of Approximation Theory - Jagiellonian University

March 8, 2023

Joint work with

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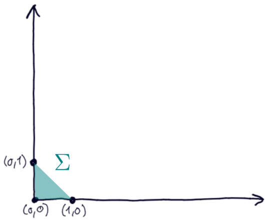
Done with the support of the Icelandic Research Fund (grant 207236-051) and the Science Institute, University of Iceland.

The unit simplex

The unit simplex in $\mathbb{R}^n$ is $\Sigma = \{x \in \mathbb{R}^n_+; x_1 + \cdots + x_n \leq 1\}$



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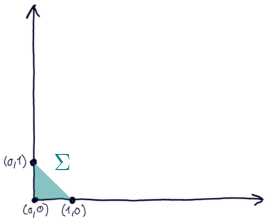

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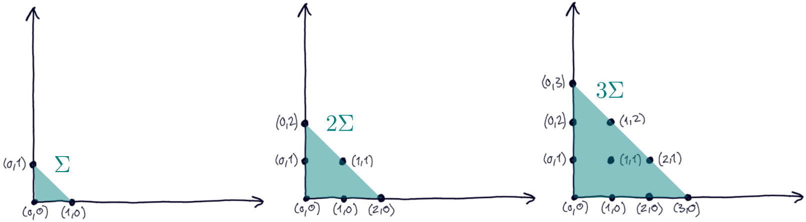

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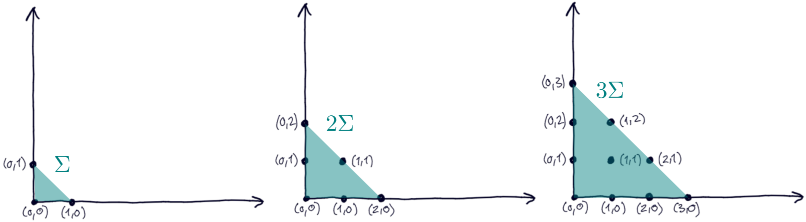

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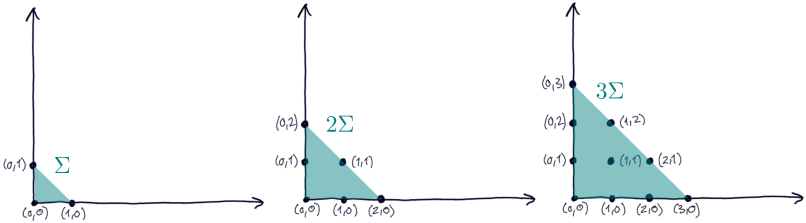

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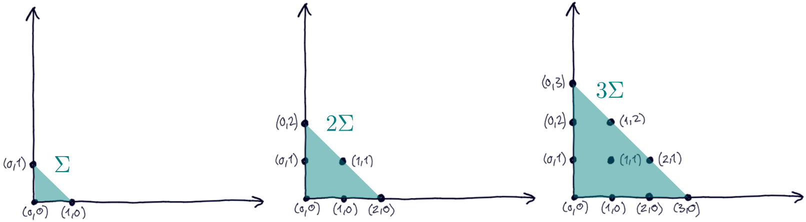

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Question: What happens when we use a different shape from Σ?
What properties of Σ are important? Neighborhood of zero, projections to the axes, symmetry, interior, ...

Polynomials with exponents in convex sets

Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. For every $m \in \mathbb{N}$ we let $\mathcal{P}^S_m(\mathbb{C}^n)$ by all polynomials in n complex variables of the form

$$p(z) = \sum_{\alpha \in (mS) \cap \mathbb{N}^n} a_\alpha z^\alpha, z \in \mathbb{C}^n$$

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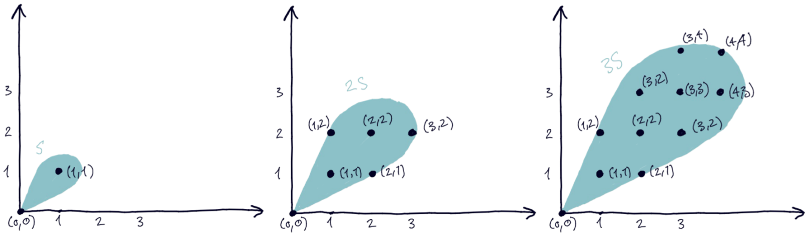

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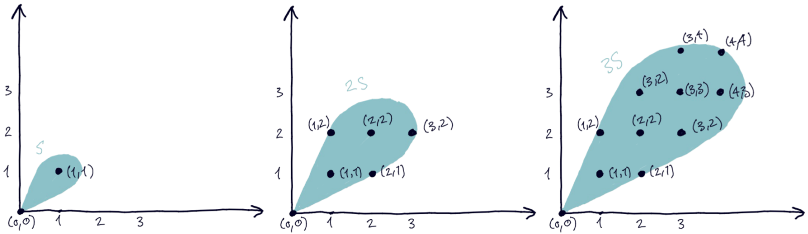

Note

This theory does not provide anything new when $n = 1$.

Our settings

We will assume $0 \in S$ and S is convex and compact.
This implies $\mathcal{P}^S(\mathbb{C}^n)$ is a graded ring, since

$$\mathcal{P}^S_j(\mathbb{C}^n)\mathcal{P}^S_k(\mathbb{C}^n) \subset \mathcal{P}^S_{j+k}(\mathbb{C}^n).$$

Supporting function

Define the *supporting function* of S as $\phi_S(\xi) = \sup_{x \in S}\langle x, \xi\rangle$, $\xi \in \mathbb{R}^n$.
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$$\phi_S(\xi) = \max_{x \in \operatorname{ext} S}\langle x, \xi\rangle, \qquad \xi \in \mathbb{R}^n$$

$$\phi_{S_1+S_2}(\xi) = \phi_{S_1}(\xi) + \phi_{S_2}(\xi)$$

$$\phi_{\lambda S(\xi)} = \lambda\phi_S(\xi)$$

Logarithmic supporting functions

For $z \in \mathbb{C}^{*n}$ we define the *logarithmic supporting function*

$$H_S(z) = (\phi_S \circ (\log|z_1|, \cdots, \log|z_n|)) = \sup_{s \in S}(s_1 \log|z_1| + \cdots + s_n \log|z_n|).$$

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Remark

$$H_S(z) \leq \phi_S(1, \ldots, 1) \log^+ \|z\|_\infty.$$

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A plurisubharmonic function u on $\Omega \subset \mathbb{C}^n$ is *maximal* if for every $G \subset\subset \Omega$ and $v \in \mathcal{USC}(\overline{G}) \cap \mathcal{PSH}(G)$ such that $v \leq u$ on ∂G implies $v \leq u$ on G.

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Maximality of H_S

H_S is maximal outside of the boundary of $\{H_S = 0\}$.

Examples

For $\Sigma \subset \mathbb{R}^n_+$ we have $\phi_\Sigma(\xi) = \max\{0, \xi_1, \ldots, \xi_n\}$ and

$$H_S(z) = \max_{j=1,\ldots,n} \log^+ |z_j| = \log^+ \|z\|_\infty.$$

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For $S = \mathrm{ch}((0,0),(1,0),(1,1)) \in \mathbb{R}^2_+$ we have $\phi_S(\xi) = \max\{0, \xi_1, \xi_1 + \xi_2\}$ and

$$H_S(z) = \max\{0, \log|z_1|, \log|z_1| + \log|z_2|\}.$$

The Lelong class with respect to S

Define the *Lelong class* $\mathcal{L}^S(\mathbb{C}^n) = \{u \in \mathcal{PSH}(\mathbb{C}^n); u(z) \leq c_u + H_S(z)\}$.

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Let $p \in \mathcal{O}(\mathbb{C}^n)$, then $p \in \mathcal{P}_m^S(\mathbb{C}^n)$ if and only if $\log |p|^{1/m} \in \mathcal{L}^S(\mathbb{C}^n)$.

The Siciak-Zakharyuta function

For $E \subset \mathbb{C}^n$ and $q : E \to \mathbb{R} \cup \{+\infty\}$ we define

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- q is lower semi-continuous ($q \in \mathcal{LSC}(\mathbb{C}^n)$),
- $\{z \in E; q(z) < +\infty\}$ is non-pluripolar, and
- if E is unbounded $\lim_{E \ni z, |z| \to \infty}(H_S(z) - q(z)) = -\infty$.

Properties of $V^S_{E,q}$

- $V^{S*}_{K,q} \in \mathcal{L}^S(\mathbb{C}^n)$ where * denotes the upper regularization.
- $V^S_{K,q} \in \mathcal{LSC}(\mathbb{C}^{*n})$, and
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Limits

Let S_j, $j \in \mathbb{N}$ and S be compact convex subsets of $\mathbb{R}^n_+$ with $0 \in S$ and $S_j \searrow S$, and q be an admissible weight on a compact subset K of $\mathbb{C}^n$.

- $\mathcal{L}^S(\mathbb{C}^n) = \cap_{j\in\mathbb{N}} \mathcal{L}^{S_j}(\mathbb{C}^n)$.
- If $V^{S_j*}_{K,q} \leq q$ on K for some j, then $V^{S_j}_{K,q} \searrow V^S_{K,q}$ as $j \to \infty$.
- If $(q_j)_{j\in\mathbb{N}}$ is a sequence $\mathcal{LSC}(K)$ and $q_j \nearrow q$, then q_j is an admissible weight for every j and $V^{S*}_{K,q} = \big(\lim_{j\to\infty} V^{S*}_{K,q_j}\big)^*$.

The Siciak extremal function

Let $E \subset \mathbb{C}^n$ and $q : E \to \mathbb{R} \cup \{+\infty\}$. For $m \in \mathbb{N}$ we define

$$\Phi^S_{E,q,m}(z) = \sup\{|p(z)|^{1/m}; p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \leq 1\},$$

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Proposition

For $j, k = 1, 2, 3, \ldots$

$$\big(\Phi^S_{E,q,j}(z)\big)^j \big(\Phi^S_{E,q,k}(z)\big)^k \leq \big(\Phi^S_{E,q,j+k}(z)\big)^{j+k}, \qquad z \in \mathbb{C}^n,$$

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If q is bounded below and $\Phi^S_{E,q}$ is continuous on some compact subset X of $\mathbb{C}^n$, then the convergence is uniform on X.

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Lower sets

The set S is a *lower set* if for a point $s \in S$ then $t \in S$ where $0 \leq t_j \leq s_j$ for $j = 1, \ldots, n$.

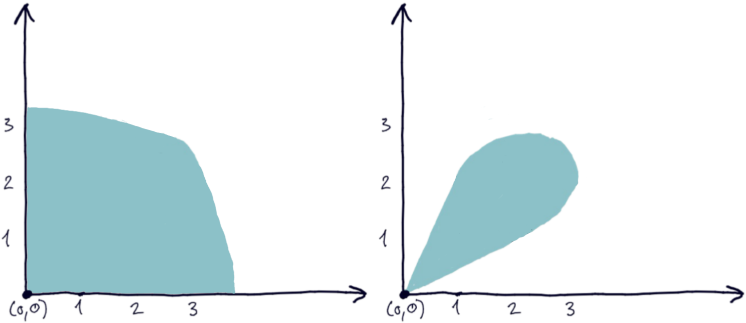

Figure: Lower set (left) and not a lower set (right)

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If $K \subset \mathbb{C}^n$ is compact and q is an admissible weight on K, then

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Theorem (Bos-Levenberg, Bayrakter et.al)

Let $0 \in S \subset \mathbb{R}^n_+$ be a compact, convex, lower set with non-empty interior. If $K \subset \mathbb{C}^n$ is closed and q an admissible weight, then

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Example

If $S = \mathrm{ch}\{(0,0),(\pi,1)\}$ then we do not have an equality above.

Product formula

With $S = \Sigma$ and $q = 0$ we have for compact sets $K_j \subset \mathbb{C}^{n_j}$ that

$$V_{K_1 \times K_2}(z) = \max\{V_{K_1}(z_1), V_{K_2}(z_2)\}, \qquad z = (z_1, z_2) \in \mathbb{C}^{n_1+n_2}.$$

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Levenberg and Perera have the following variant of this: Let $K_1, \dots, K_n$ be compact subsets of $\mathbb{C}$ and S a lower set, then

$$V_{K_1 \times \cdots \times K_n}(z) = \phi_S(V^*_{K_1}(z_1), \dots, V^*_{K_n}(z_n)).$$

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Example

The following example shows that the lower set requirement are necessary. Let $K_1 = K_2 = \overline{\mathbb{D}}$, then $V_{K_j}(z_j) = \log^+ |z_j|$. Let $S = \mathrm{ch}\{(0,0), (1,0), (1,1), (0,a)\}$, then

$$\phi_S = \max\{0, \xi_1, \xi_1 + \xi_2, a\xi_2\}.$$

However

$$\phi_S(V_{\overline{\mathrm{D}}}(z_1), V_{\overline{\mathrm{D}}}(z_1)) = \phi_S(\xi^+)$$

Theorem

Let S be a compact convex subset of $\mathbb{R}^n_+$, $0 \in S$, $m \in \mathbb{N}^*$, and $d_m = d(mS, \mathbb{N}^n \setminus mS)$ denote the euclidean distance between the sets mS and $\mathbb{N}^n \setminus mS$. Let $f \in \mathcal{O}(\mathbb{C}^n)$, assume that

$$\int_{\mathbb{C}^n} |f|^2(1+|\zeta|^2)^{-\gamma} e^{-2mH_S}\, d\lambda < +\infty$$

for some $0 \le \gamma < d_m$, and let γ_0 denote the infimum of such γ. Let Γ be the cone consisting of all ξ such that the angle between the vectors $1 = (1, \dots, 1)$ and ξ is $\le \arccos(-(d_m - \gamma_0)/\sqrt{n})$ and let $\widehat{S}_\Gamma$ be the hull of S with respect to the cone Γ defined by

$$\hat{S}_\Gamma = \{x \in \mathbb{R}^n_+; \langle x, \xi\rangle \le \phi_S(\xi), \forall \xi \in \Gamma\}.$$

Then $f \in \mathcal{P}_m^{\widehat{S}_\Gamma}(\mathbb{C}^n)$.

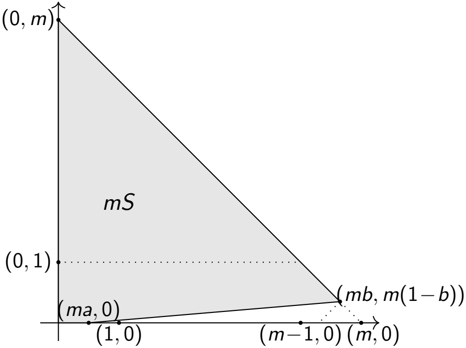
Corollary

If in addition S is a lower set then $f \in \mathcal{P}_m^S(\mathbb{C}^n)$.

Example

Fix m and let $0 < a < b < 1$ and define $S \subseteq \mathbb{R}^2_+$ as the quadrangle

$$S = \mathrm{ch}\{(0,0),(a,0),(b,1-b),(0,1)\}.$$



For a small enough and b close enough to 1 we can show that $f(z) = z_1^k$, $k = 1, \ldots, m-1$ satisfy the previous L^2 estimate, but they are clearly not in $\mathcal{P}_m^S(\mathbb{C}^2)$.

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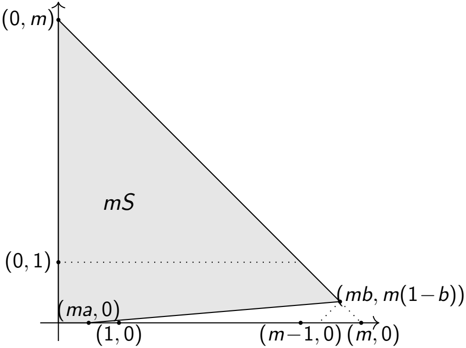

For a small enough and b close enough to 1 we can show that $f(z) = z_1^k$, $k = 1, \dots, m-1$ satisfy the previous L^2 estimate, but they are clearly not in $\mathcal{P}_m^S(\mathbb{C}^2)$. This shows that it is necessary to use the hull of S in the Theorem above.

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- We have a product formula for V_K^S when S in an lower set.
- We (at least) have a Siciak-Zakharjuta theorem when S is an lower set. Definitely not always.
- Both $V_{K,q}^S$ and $\Phi_{K,q}^S$ have similar properties as $V_{K,q}$ and $\Phi_{K,q}$.
- (Not shown here) We can connect $V_{K,q}^S$ to polynomials approximations with $\mathcal{P}^S(\mathbb{C}^n)$, i.e. a Bernstein-Walsh theorem.

Thanks